Solutions Linear Algebra I exam, Feb 2012

1. (a) $\underset{\sim}{x}=\underset{\sim}{0}$ is always a solution of $A\underset{\sim}{x}=\underset{\sim}{0}$.
Hence: For all $\beta\in\mathbb{R}$, $A\underset{\sim}{x}=\underset{\sim}{0}$ is consistent.

(b)
$$A=\begin{pmatrix}1&2&0\\2&3&1\\1&1&\beta\end{pmatrix}\xrightarrow[\mathrm{III}\to\mathrm{III}-\mathrm{I}]{\mathrm{II}\to\mathrm{II}-2\mathrm{I}}\begin{pmatrix}1&2&0\\0&-1&1\\0&-1&\beta\end{pmatrix}\xrightarrow{\mathrm{III}\to\mathrm{III}-\mathrm{II}}\begin{pmatrix}1&2&0\\0&-1&1\\0&0&\beta-1\end{pmatrix}$$

For $\beta\neq1$, there is exactly one solution, namely $\underset{\sim}{x}=\underset{\sim}{0}$

(c) For $\beta=1$, there are infinitely many solutions:
$x_3\in\mathbb{R}$, $x_2=x_3$, $x_1=-2x_2=-2x_3$
$\Rightarrow$ Solution set $=\operatorname{span}\left(\begin{pmatrix}-2\\1\\1\end{pmatrix}\right)$.

(d) augmented matrix
$$(A|\underset{\sim}{b})=\left(\begin{array}{ccc|c}1&2&0&0\\2&3&1&0\\1&1&\beta&1\end{array}\right)\xrightarrow[\mathrm{III}\to\mathrm{III}-\mathrm{I}]{\mathrm{II}\to\mathrm{II}-2\mathrm{I}}\left(\begin{array}{ccc|c}1&2&0&0\\0&-1&1&0\\0&-1&\beta&1\end{array}\right)\xrightarrow{\mathrm{III}\to\mathrm{III}-\mathrm{II}}\left(\begin{array}{ccc|c}1&2&0&0\\0&-1&1&0\\0&0&\beta-1&1\end{array}\right)$$

For $\beta=1$, the system is inconsistent.

(e) For $\beta\neq1$, $x_3=1/\beta-1$, $x_2=x_3=1/\beta-1$, $x_1=-2x_2=\frac{2}{1-\beta}$

Solution set $=\left\{\frac{1}{\beta-1}\begin{pmatrix}-2\\1\\1\end{pmatrix}\right\}$

2. (a) To be shown: $\underset{\sim}{x},\underset{\sim}{y}\in N(A)$, $\alpha\in\mathbb{R}\Rightarrow$ (i) $\underset{\sim}{x}+\underset{\sim}{y}\in N(A)$
(ii) $\alpha\underset{\sim}{x}\in N(A)$

(i) $\underset{\sim}{x},\underset{\sim}{y}\in N(A)\Rightarrow A\underset{\sim}{x}=A\underset{\sim}{y}=\underset{\sim}{0}\Rightarrow A\underset{\sim}{x}+A\underset{\sim}{y}=\underset{\sim}{0}$
$\Rightarrow A(\underset{\sim}{x}+\underset{\sim}{y})=\underset{\sim}{0}\Rightarrow\underset{\sim}{x}+\underset{\sim}{y}\in N(A)$

(ii) $A\underset{\sim}{x}=\underset{\sim}{0}\Rightarrow\alpha A\underset{\sim}{x}=\alpha\underset{\sim}{0}\Rightarrow A(\alpha\underset{\sim}{x})=\underset{\sim}{0}\Rightarrow\alpha\underset{\sim}{x}\in N(A)$.

(b) To be shown: $K\neq\emptyset\Leftrightarrow\underset{\sim}{b}\in R(A)$
$K\neq\emptyset\Leftrightarrow\exists\,\underset{\sim}{x}\in K\Leftrightarrow\exists\,\underset{\sim}{x}\in\mathbb{R}^n$ such that $A\underset{\sim}{x}=\underset{\sim}{b}$
$\Leftrightarrow\exists\,\underset{\sim}{x}\in\mathbb{R}^n$ s.t. $x_1\underset{\sim}{a}_1+\dots+x_n\underset{\sim}{a}_n=\underset{\sim}{b}$ where $\underset{\sim}{a}_1,\dots,\underset{\sim}{a}_n$ column vectors of A
$\Leftrightarrow\underset{\sim}{b}$ linear combination of column vectors of A
$\Leftrightarrow\underset{\sim}{b}\in R(A)$

(c) $\underset{\sim}{v} \in K$. To be shown: $K = \{ \underset{\sim}{v} + \underset{\sim}{w} \mid \underset{\sim}{w} \in N(A) \}$

"$\subset$": Let $\underset{\sim}{x} \in K \Rightarrow A\underset{\sim}{x} = A\underset{\sim}{v} = \underset{\sim}{b}$

$\Rightarrow A(\underset{\sim}{x} - \underset{\sim}{v}) = \underset{\sim}{0}$

$\Rightarrow \underset{\sim}{w}' := \underset{\sim}{x} - \underset{\sim}{v} \in N(A)$

$\Rightarrow A(\underset{\sim}{v} + \underset{\sim}{w}') = Av + Aw' = \underset{\sim}{b} + \underset{\sim}{0} = \underset{\sim}{b}$

$\Rightarrow \underset{\sim}{x} = \underset{\sim}{v} + \underset{\sim}{w}' \in \{ \underset{\sim}{v} + \underset{\sim}{w} \mid \underset{\sim}{w} \in N(A) \}$

"$\supset$": Let $\underset{\sim}{x} \in \{ \underset{\sim}{v} + \underset{\sim}{w} \mid w \in N(A) \}$

$\Rightarrow \exists \underset{\sim}{w} \in N(A)$ with $\underset{\sim}{x} = \underset{\sim}{v} + \underset{\sim}{w}$

$\Rightarrow A\underset{\sim}{x} = A(\underset{\sim}{v} + \underset{\sim}{w}) = A\underset{\sim}{v} + A\underset{\sim}{w}$

$= \underset{\sim}{b} + \underset{\sim}{0} = \underset{\sim}{b}$

$\Rightarrow \underset{\sim}{x} \in K$.

(d) For $K \subset \mathbb{R}^n$ linear subspace we need (i) $\underset{\sim}{x}, \underset{\sim}{y} \in K \Rightarrow \underset{\sim}{x} + \underset{\sim}{y} \in K$

(ii) $\underset{\sim}{x} \in K, \alpha \in \mathbb{R} \Rightarrow \alpha \underset{\sim}{x} \in K$

First case $K = \emptyset$ ✓

Second case $K \neq \emptyset$: ~~[illegible]~~ $\exists \underset{\sim}{v} \in K$

From (c) we know $K = \{ \underset{\sim}{v} + \underset{\sim}{w} \mid \underset{\sim}{w} \in N(A) \}$

Let $\underset{\sim}{x}, \underset{\sim}{y} \in K \Rightarrow \exists \underset{\sim}{w}_1, \underset{\sim}{w}_2 \in N(A)$ with $\underset{\sim}{x} = \underset{\sim}{v} + \underset{\sim}{w}_1$

$\underset{\sim}{y} = \underset{\sim}{v} + \underset{\sim}{w}_2$

$\Rightarrow A(\underset{\sim}{x} + \underset{\sim}{y}) = A(2\underset{\sim}{v} + \underset{\sim}{w}_1 + \underset{\sim}{w}_2)$

$= 2A\underset{\sim}{v} + A\underset{\sim}{w}_1 + A\underset{\sim}{w}_2$

$= 2\underset{\sim}{b}$

The latter is different $\underset{\sim}{b}$ unless $\underset{\sim}{b} = \underset{\sim}{0}$.

For $\underset{\sim}{b} = \underset{\sim}{0}$ also (ii) holds (see (a))

Hence: K lin. subspace iff $\underset{\sim}{b} = \underset{\sim}{0}$ or $K = \emptyset$.

3. (a) A, B nonsingular $\Rightarrow$ dim rowspace A = dim row space $B = n$

$\Rightarrow$ dim rowspace $M = 2n$ (consider linear combination of row vectors of M)

(b) A, B nonsingular. Ansatz $M^{-1} := \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}$

$$\Rightarrow M^{-1} M = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} \begin{pmatrix} A & I \\ 0 & B \end{pmatrix} = \begin{pmatrix} M_1 A & M_1 + M_2 B \\ M_3 A & M_3 + M_4 B \end{pmatrix} \overset{!}{=} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

$$\Rightarrow M_1 = A^{-1}, M_3 = 0, M_4 = B^{-1}$$

$$A^{-1} + M_2 B = 0 \Rightarrow M_2 = -A^{-1} B^{-1}$$

$$\Rightarrow M^{-1} = \begin{pmatrix} A^{-1} & -A^{-1} B^{-1} \\ 0 & B^{-1} \end{pmatrix}$$

Check also: $MM^{-1} = \begin{pmatrix} A & I \\ 0 & B \end{pmatrix} \begin{pmatrix} A^{-1} & -A^{-1} B^{-1} \\ 0 & B^{-1} \end{pmatrix} = \begin{pmatrix} AA^{-1} & -AA^{-1}B^{-1} + B^{-1} \\ 0 & BB^{-1} \end{pmatrix}$

$$= \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

(c) $\begin{pmatrix} A & I \\ 0 & B \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} \Rightarrow \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} A & I \\ 0 & B \end{pmatrix}^{-1} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} A^{-1} & -A^{-1} B^{-1} \\ 0 & B^{-1} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$

$$= \begin{pmatrix} A^{-1} A - A^{-1} B^{-1} B \\ B^{-1} B \end{pmatrix} = \begin{pmatrix} I - A^{-1} \\ I \end{pmatrix}$$

4. (a) $p, q \in P_3, \alpha \in \mathbb{R}$. To be shown: (i) $T(p+q) = T(p) + T(q)$

(ii) $T(\alpha p) = \alpha T(p)$

(i) $(p+q)(x) = p(x) + q(x) \quad \forall x \in \mathbb{R}$

$$\Rightarrow T(p+q) = \begin{pmatrix} (p+q)(1) \\ (p+q)(2) \\ (p+q)(3) \end{pmatrix} = \begin{pmatrix} p(1) + q(1) \\ p(2) + q(2) \\ p(3) + q(3) \end{pmatrix} = \begin{pmatrix} p(1) \\ p(2) \\ p(3) \end{pmatrix} + \begin{pmatrix} q(1) \\ q(2) \\ q(3) \end{pmatrix} = T(p) + T(q)$$

(ii) $(\alpha p)(x) = \alpha p(x) \quad \forall x \in \mathbb{R}$

$$\Rightarrow T(\alpha p) = \begin{pmatrix} (\alpha p)(1) \\ (\alpha p)(2) \\ (\alpha p)(3) \end{pmatrix} = \begin{pmatrix} \alpha p(1) \\ \alpha p(2) \\ \alpha p(3) \end{pmatrix} = \alpha \begin{pmatrix} p(1) \\ p(2) \\ p(3) \end{pmatrix} = \alpha T(p)$$

(b) $p \in P_3 \Rightarrow p = a + bx + cx^2$, $a, b, c \in \mathbb{R}$

$$T(p) = \begin{pmatrix} a + b + c \\ a + 2b + 4c \\ a + 3b + 9c \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}}_{=: A} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

(c)
$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} \xrightarrow[\text{III} \to \text{III} - \text{I}]{\text{II} \to \text{II} - \text{I}} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 8 \end{pmatrix} \xrightarrow{\text{III} \to \text{III} - 2\,\text{II}} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{pmatrix}$$

$\Rightarrow$ A has rank 3

(d) $\dim \ker(A) + \dim R(A) = 3$

$\Rightarrow \dim \ker(A) = 3 - 3 = 0$

(e) $N(A) = \{\underset{\sim}{0}\} \Rightarrow p \in \ker(T) \Leftrightarrow p(x) = a + bx + cx^2$ with $a = b = c = 0$

$\Leftrightarrow p = 0$ (the zero polynomial)

5. (a) To be shown: $A^T = A$

$$A^T = (\underset{\sim}{x}\underset{\sim}{y}^T + \underset{\sim}{y}\underset{\sim}{x}^T)^T = (\underset{\sim}{x}\underset{\sim}{y}^T)^T + (\underset{\sim}{y}\underset{\sim}{x}^T)^T = (\underset{\sim}{y}^T)^T \underset{\sim}{x}^T + (\underset{\sim}{x}^T)^T \underset{\sim}{y}^T$$
$$= \underset{\sim}{y}\underset{\sim}{x}^T + \underset{\sim}{x}\underset{\sim}{y}^T = \underset{\sim}{x}\underset{\sim}{y}^T + \underset{\sim}{y}\underset{\sim}{x}^T = A$$

(b) $\underset{\sim}{z} \in N(A) \Leftrightarrow (\underset{\sim}{x}\underset{\sim}{y}^T + \underset{\sim}{y}\underset{\sim}{x}^T)\underset{\sim}{z} = \underset{\sim}{0}$

$N(A) \subseteq S^\perp$:

$\Leftrightarrow (\underset{\sim}{x}\underset{\sim}{y}^T)\underset{\sim}{z} + (\underset{\sim}{y}\underset{\sim}{x}^T)\underset{\sim}{z} = \underset{\sim}{0}$

$\Leftrightarrow \underset{\sim}{x}(\underset{\sim}{y}^T\underset{\sim}{z}) + \underset{\sim}{y}(\underset{\sim}{x}^T\underset{\sim}{z}) = \underset{\sim}{0}$

$\Rightarrow \alpha\underset{\sim}{x} + \beta\underset{\sim}{y} = 0$ with $\alpha = \underset{\sim}{y}^T\underset{\sim}{z}$, $\beta = \underset{\sim}{x}^T\underset{\sim}{z}$

$\Rightarrow \alpha = \beta = 0$ because $\underset{\sim}{x}, \underset{\sim}{y}$ linearly independent

$\Rightarrow \underset{\sim}{y}^T\underset{\sim}{z} = 0$ and $\underset{\sim}{x}^T\underset{\sim}{z} = 0$

$\Rightarrow \underset{\sim}{x} \perp \underset{\sim}{z}$ and $\underset{\sim}{y} \perp \underset{\sim}{z}$

$\Rightarrow \underset{\sim}{z} \in S^\perp$

$S^\perp \subseteq N(A)$: reverse arguments above.

(c) $N(A) = S^\perp$, $S \oplus S^\perp = \mathbb{R}^3$

$\Rightarrow \dim N(A) = \dim S^\perp = \dim \mathbb{R}^3 - \dim S = 3 - 2 = 1$

Moreover:

$\dim N(A) + \operatorname{rank} A = \dim \mathbb{R}^3$ (Rank-Nullity-Th^{m})

$\Rightarrow \operatorname{rank} A = 3 - 1 = 2$

6. (a) $p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 4-\lambda & -5 & 1 \\ 1 & -\lambda & -1 \\ 0 & 1 & -1-\lambda \end{vmatrix} = \lambda(1-\lambda)(\lambda-2)$

(b) $p(\lambda) = 0 \Leftrightarrow \lambda = 0 \lor \lambda = 1 \lor \lambda = 2$

(c) eigenvectors:

$\lambda = 0$: $A\underset{\sim}{x} = \underset{\sim}{0}$ has solution $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\lambda = 1$: $(A - I)\underset{\sim}{x} = \underset{\sim}{0}$ — " — $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$

$\lambda = 2$: $(A - 2I)\underset{\sim}{x} = \underset{\sim}{0}$ — " — $\begin{pmatrix} 7 \\ 3 \\ 1 \end{pmatrix}$

(d) $X := \begin{pmatrix} 1 & 3 & 7 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}$, $X^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -4 & 5 \\ -2 & 6 & -4 \\ 1 & -2 & 1 \end{pmatrix}$

$\Rightarrow X^{-1} A X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.